Fast Algorithms for Image Reconstruction from Very Few Data

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Recent developments in applied mathematics have demonstrated that images and other signals can be reconstructed from far fewer data than traditionally believed necessary [1,2]. The results exploit the fact that images of the real world or human-generated experiments are a very special subset of all possible images, the overwhelming majority of which will look like random noise. When we try to reconstruct an image from data, what sets apart the image we are looking for is that it is sparse – it can be represented with relatively few coefficients, far fewer than the number of pixels in the image. We have developed reconstruction algorithms that can reconstruct images from a number of data that is commensurate with the complexity (or information content) of the image, rather than its size. These algorithms are at the forefront of the new and popular field known as compressive sensing, so called because the few data from which an image can be effectively reconstructed constitute a compression of an image. Instead of needing to obtain every pixel of an image and then compressing it (as is done by digital cameras, for example), we can, in effect, measure a compression directly.

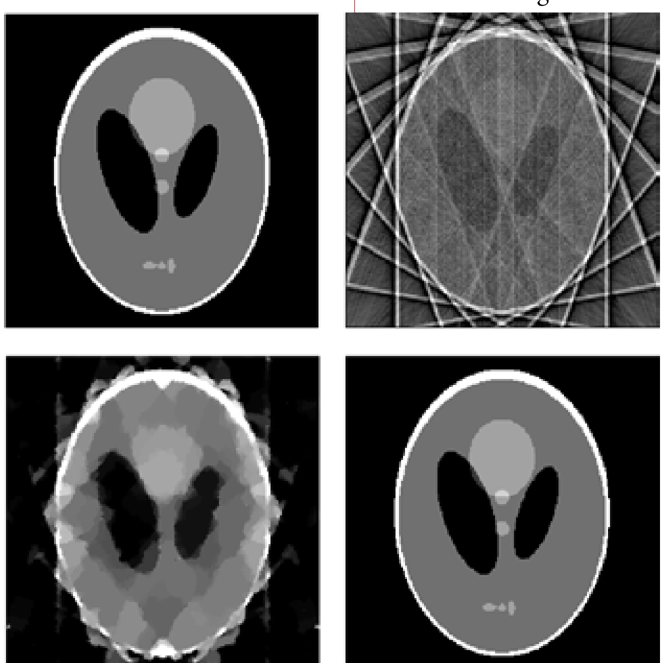

Fig. 1. Top left: the Shepp-Logan phantom, a test image. Top right: reconstruction using filtered back-projection is poor. Bottom left: reconstruction using usual, convex compressive sensing method is also poor. Bottom right: our nonconvex reconstruction is exact.

There are many applications of the ability to reconstruct images from very few data, particularly whenever data are difficult, expensive, or dangerous to acquire. For example, in medical imaging the high X-ray dose of a CT scan can be replaced with relatively few radiographs. Better yet, in many cases the much safer MRI can be substituted—the usual barrier is the high cost of the procedure, due to the long scanning time required. Compressive sensing techniques can allow a much shorter scan to be used instead, making the procedure more palatable to insurance companies. Both CT and MRI are also used in national security applications, where the ability to reduce data collection time can be very advantageous. Many other applications related to LANL's mission stand to benefit as well. In remote sensing or space situational awareness, one is often fundamentally limited in the amount of information one can gather. Compressive sensing would allow more information to be extracted from the data that are available.

As a simple example, consider Fig. 1. We reconstruct a test image, the Shepp-Logan phantom, which was designed to be challenging for medical imaging algorithms to reconstruct. Our data are samples of the 2D Fourier transform of the phantom, taken along nine radial lines through the origin of the frequency domain, together making up less than 3.5% of the full Fourier transform. Sampling along radial lines in this way makes the data roughly equivalent to having radiographs of the phantom, one for each line. The reconstruction problem can thus be seen as a limited-view CT problem, but also as an MRI problem, as MRI data can be seen as directly sampling the Fourier transform as the object. Our reconstruction takes advantage of the fact that the gradient of the phantom is very sparse, being zero except at the boundaries of the ellipses. Our reconstruction approach is to solve an optimization problem, which minimizes a sparsity-inducing penalty term, subject to the data constraint. The usual reconstruction method for CT is filtered backprojection, which gives a very poor reconstruction, being designed for having hundreds of radiographs. The usual compressive sensing approach also fails with so few data, but our particular method gives an exact reconstruction.

The reason our compressive sensing approach outperforms the usual one is that it is a closer approximation to what we really want to solve, namely the problem of simply finding the sparsest solution that is consistent with the data. However, directly solving that problem is computationally intractable; all the world's computers working for trillions of years would hardly make a dent. The field of compressive sensing was born when it was discovered that one could replace this problem with one that is convex. Convexity of a function implies that any local minimum will be a global minimum—this means, for example, that one can just head downhill, such as moving in the direction opposite of the gradient of the function, and eventually get to a global minimum. The research field of convex optimization is very mature, so there are several computationally efficient algorithms available for solving the convex approximation of the sparse recovery problem. Under reasonable conditions, it has been proven that the solution of the convex problem will be the same as that of the intractable sparse recovery problem. This result generated substantial excitement, with compressive sensing now being one of the fastest growing areas of applied mathematics.

Our approach, however, is nonconvex. Simply heading downhill will almost certainly result in convergence to a local minimum, but not the global minimum. For the Shepp-Logan phantom example, the number of local minima exceeds $10^{4,500}$! This makes it seemingly impossible to hope to obtain the global minimum. However, we have developed an algorithm approach that does so, with tremendous reliability [3,4]. The key is an iterative smoothing approach, which first smooths out the function being minimized so that it no longer has any local minima. The smoothing is then successively diminished, with the solution at each stage used to initialize the next stage. The result is that the iteration reaches the right convergence basin, before the local minima reappear. We thus obtain algorithms that are able to avoid the local minimum problem, and also serve to more closely approximate the true sparse recovery problem. This is what allows our algorithms to successfully reconstruct images from many fewer data than the usual, convex approach to compressive sensing.

Our latest research has led to an algorithm that not only works with fewer data than its predecessors, but is computationally very fast, at least for many image reconstruction problems [5]. It works particularly well for cases where the data consist of samples of the Fourier transform, as in the example above. In addition to CT and MRI, Fourier-domain sampling arises in applications such as synthetic aperture radar and sonar, or where interferometry is used, such as in radio-astronomy or spectroscopy. Most algorithms that converge in a reasonable number of iterations require solving a large, linear system during each iteration. These can be computationally expensive to solve, especially for large-scale problems. Our approach allows this system to be solved directly in the Fourier domain, essentially requiring just two Fast Fourier Transforms (FFTs). Using FFTs makes solving the linear system much faster, while also scaling very well to large problems. For example, the first Shepp-Logan phantom experiments took the better part of an hour to reconstruct the 256×256 image. The new method takes just seconds, using very simple prototype code (in Matlab) on a simple laptop (see Fig.2). Thus, using nonconvex optimization, with unparalleled abilities to recover images from fewer data than ever before, image reconstruction can now be done using algorithms that are efficient, making many large-scale applications much more feasible.

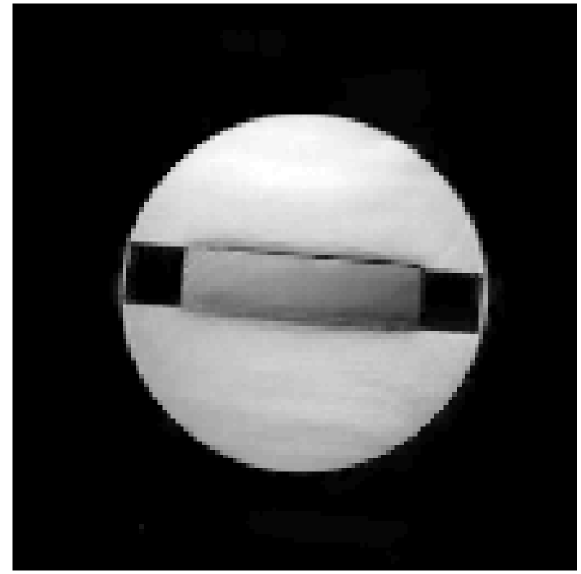

Fig. 2. Reconstruction of a real test object from 8-coil MRI data with 31% sampling. All eight reconstructions were performed in 17 s, using simple Matlab code on a laptop.

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